

# On a Class of Two-Dimensional Singular Douglas and Projectively flat Finsler Metrics

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## Abstract

Singular Finsler metrics, such as Kropina metrics and  $m$ -Kropina metrics, have a lot of applications in the real world. In this paper, we study a class of two-dimensional singular Finsler metrics defined by a Riemann metric  $\alpha$  and 1-form  $\beta$ , and we characterize those which are Douglasian or locally projectively flat by some equations. It shows that the main class induced is an  $m$ -Kropina metric plus a linear part on  $\beta$ . For this class, the local structure of Douglasian or (in part) projectively flat case is determined, and in particular we show that a Kropina metric is always Douglasian and a Douglas  $m$ -Kropina metric with  $m \neq -1$  is locally Minkowskian. It indicates that the two-dimensional case is quite different from the higher dimensional ones.

**Keywords:**  $(\alpha, \beta)$ -Metric,  $m$ -Kropina Metric, Douglas Metric, Projectively Flat  
**MR(2000) subject classification:** 53A20, 53B40

## 1 Introduction

There are two important projective invariants in projective Finsler geometry: the Douglas curvature ( $\mathbf{D}$ ) and the Weyl curvature ( $\mathbf{W}^o$  in dimension two and  $\mathbf{W}$  in higher dimensions) ([5]). A Finsler metric is called *Douglasian* if  $\mathbf{D} = 0$ . Roughly speaking, a Douglas metric is a Finsler metric having the same geodesics as a Riemannian metric. A Finsler metric is said to be *locally projectively flat* if at every point, there are local coordinate systems in which geodesics are straight. As we know, the locally projectively flat class of Riemannian metrics is very limited, nothing but the class of constant sectional curvature (Beltrami Theorem). However, the class of locally projectively flat Finsler metrics is very rich. Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics, and meanwhile there are many Douglas metrics which are not locally projectively flat.

In this paper, we will concentrate on a special class of two-dimensional Finsler metrics:  $(\alpha, \beta)$ -metrics, and characterize those which are Douglasian and locally projectively flat under the condition (2) below. An  $(\alpha, \beta)$ -metric is defined by a Riemannian metric  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  and a 1-form  $\beta = b_i(x)y^i$  on a manifold  $M$ , which can be expressed in the following form:

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where  $\phi(s)$  is a function satisfying certain conditions. It is known that  $F$  is a regular Finsler metric if  $\beta$  satisfies  $\|\beta\|_\alpha < b_o$  and  $\phi(s)$  is  $C^\infty$  on  $(-b_o, b_o)$  satisfying

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad (|s| \leq \rho < b_o), \quad (1)$$

where  $b_o$  is a positive constant ([13]). If  $\phi(0)$  is not defined or  $\phi$  does not satisfy (1), then the  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  is singular. Singular Finsler metrics have a lot of applications in the real world ([1] [2]). Z. Shen also introduces singular Finsler metrics in [14].

Assume that  $\phi(s)$  is in the following form

$$\phi(s) := cs + s^m \varphi(s), \quad (2)$$

where  $c, m$  are constant with  $m \neq 0, 1$  and  $\varphi(s)$  is a  $C^\infty$  function on a neighborhood of  $s = 0$  with  $\varphi(0) = 1$ , and further for convenience we put  $c = 0$  if  $m$  is a negative integer. If  $m = 0$ , we have  $\phi(0) = 1$  and this case appears in a lot of literatures. When  $m \geq 2$  is an integer, (2) is equivalent to the following condition

$$\phi(0) = 0, \quad \phi^{(k)}(0) = 0 \quad (2 \leq k \leq m-1), \quad \phi^{(m)}(0) = m!.$$

Another interesting case is  $c = 0$  and  $\varphi(s) \equiv 1$  in (2), and in this case,  $F = \alpha\phi(s)$  is called an  $m$ -Kropina metric, and in particular a Kropina metric when  $m = -1$ .

The case  $\phi(0) = 1$  has been studied in a lot of interesting research papers ([6]–[8], [10] [12] [13] [19]–[21]). In [6] [12], the authors respectively study and characterize Douglas  $(\alpha, \beta)$ -metrics and locally projectively flat  $(\alpha, \beta)$ -metrics in dimensions  $n \geq 3$  and  $\phi(0) = 1$ , and further, the present author solves the case  $n = 2$  and shows that the two-dimensional case is quite different from the higher dimensional ones ([19]). In singular case, there are some papers on the studies of  $m$ -Kropina metrics and Kropina metrics ([11] [15] [16] [22]). Further, in [17], the present author classifies a class of higher dimensional singular  $(\alpha, \beta)$ -metrics  $F = \alpha\phi(\beta/\alpha)$  which are Douglasian and locally projectively flat respectively, where  $\phi(s)$  satisfies the condition (2). In this paper we will solve the singular case under the condition (2) in two-dimensional case, which shows below that the singular case is quite different from the regular condition  $\phi(0) = 1$  (cf. [19]).

**Theorem 1.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a two-dimensional  $(\alpha, \beta)$ -metric on an open subset  $U \subset \mathbb{R}^2$ , where  $\phi$  satisfies (2). Suppose  $db \neq 0$  in  $U$  and that  $\beta$  is not parallel with respect to  $\alpha$ . If  $F$  is a Douglas metric, or locally projectively flat, then  $F$  must be in the following form*

$$F = c\bar{\beta} + \bar{\beta}^m \bar{\alpha}^{1-m}, \quad (\bar{\alpha} := \sqrt{\alpha^2 + k\beta^2}, \quad \bar{\beta} := \beta), \quad (3)$$

where  $c, k$  are constant. Note that  $\bar{\alpha}$  is Riemannian if  $k > -1/b^2$ .

If  $b = \text{constant}$  in Theorem 1.1, there are other classes for the metric  $F$  (see Theorem 4.1 and Theorem 5.1 below). Theorem 1.1 also holds if  $n \geq 3$ , but there is much difference between  $n = 2$  and  $n \geq 3$  when we determine the local structures of  $F$  in (3) which is Douglasian or locally projectively flat (cf. [17]).

Theorem 1.1 naturally induces an important class of singular Finsler metric— $m$ -Kropina metric  $F = \beta^m \alpha^{1-m}$ . When  $m = -1$ ,  $F = \alpha^2/\beta$  is called a Kropina metric. There have been some research papers on Kropina metrics ([11] [16] [22]). In [15], the present author and Z. Shen characterize  $m$ -Kropina metrics which are weakly Einsteinian.

Next we determine the local structure of the metric  $F = c\beta + \beta^m \alpha^{1-m}$  which is Douglasian and locally projectively flat respectively. For projectively flat case, it is hard to deal with  $m = -1$  and  $c \neq 0, m = -3$ . The method is to apply an interesting deformation on  $\alpha$  and  $\beta$  which is defined by

$$\tilde{\alpha} := b^m \alpha, \quad \tilde{\beta} := b^{m-1} \beta, \quad (4)$$

The deformation (4) first appears in [15] for the research on weakly Einstein  $m$ -Kropina metrics. It also appears in [17]. It is very useful for  $m$ -Kropina metrics. Obviously, if  $F$  is an  $m$ -Kropina metric, then  $F$  keeps formally unchanged, namely,

$$F = \beta^m \alpha^{1-m} = \tilde{\beta}^m \tilde{\alpha}^{1-m}.$$

Further,  $\tilde{\beta}$  has unit length with respect to  $\tilde{\alpha}$ , that is,  $\|\tilde{\beta}\|_{\tilde{\alpha}} = 1$ .

**Theorem 1.2** Let  $F = c\beta + \beta^m \alpha^{1-m}$  be a two-dimensional Douglas  $(\alpha, \beta)$ -metric, where  $c, m$  are constant with  $m \neq 0, 1$ . Then we have the following cases:

- (i) ( $m = -1$ )  $\alpha$  and  $\beta$  can be arbitrary, namely, a two-dimensional metric  $F = c\beta + \alpha^2/\beta$  is always a Douglas metric.
- (ii) ( $m = -3$ )  $\alpha$  and  $\beta$  can be locally written as

$$\alpha^2 = \frac{B^3}{u^2 + v^2} \{(y^1)^2 + (y^2)^2\} - \frac{3(5 + 3cB^2)(1 + cB^2)}{B} \beta^2, \quad (5)$$

$$\beta = \frac{B^2}{(4 + 3cB^2)(u^2 + v^2)} (uy^1 + vy^2), \quad (6)$$

where  $B = B(x) > 0, u = u(x), v = v(x)$  are scalar functions such that

$$f(z) = u + iv, \quad z = x^1 + ix^2 \quad (7)$$

is a complex analytic function.

- (iii) ( $m \neq -1; c = 0$ )  $F$  can be written as  $F = \tilde{\alpha}^{1-m} \tilde{\beta}^m$ , where  $\tilde{\alpha}$  is flat and  $\tilde{\beta}$  is parallel with respect to  $\tilde{\alpha}$ , and thus  $\tilde{\alpha}$  and  $\tilde{\beta}$  can be locally written as

$$\tilde{\alpha} = |y|, \quad \tilde{\beta} = y^1. \quad (8)$$

Further  $\alpha, \beta$  are related with  $\tilde{\alpha}, \tilde{\beta}$  by

$$\alpha = \eta^{\frac{m}{m-1}} \tilde{\alpha}, \quad \beta = \eta \tilde{\beta}, \quad (9)$$

where  $\eta = \eta(x) > 0$  is a scalar function. Obviously,  $F$  is locally Minkowskian.

- (iv) ( $m \neq -1; c \neq 0, m \neq -3$ )  $F$  can be written as  $F = c\eta\tilde{\beta} + \tilde{\beta}^m \tilde{\alpha}^{1-m}$ , where (8) and (9) hold with  $\eta = \eta(x^1) > 0$ .

In Theorem 1.2, we cannot give a detailed description for the local structures of  $F$  in higher dimensions since  $\tilde{\alpha}$  cannot be determined in this case, and Theorem 1.2(i) does not hold in higher dimensions either (cf. [17]). Theorem 1.2(ii) and (iii) give two representations for the local structure of  $F = \alpha^4/\beta^3$ . We will prove Theorem 1.2(ii)-(iv) by aid of the result in [20] (also see [21]). In Theorem 1.2(ii), the metric is determined by the triple parametric functions  $B, u, v$ , where  $u, v$  are a pair of conjugate harmonious functions.

**Theorem 1.3** Let  $F = c\beta + \beta^m \alpha^{1-m}$  be a two-dimensional  $(\alpha, \beta)$ -metric, where  $c, m$  are constant with  $m \neq 0, \pm 1$  and  $c = 0$  if  $m = -3$ . Then  $F$  is Douglasian if and only if  $F$  is locally projectively flat. In this case,  $F$  is Berwaldian, or locally Minkowskian if and only if  $c = 0$  or  $\eta = \text{constant}$  in (9); and here  $\eta = \text{constant}$  implies  $\alpha$  is flat and  $\beta$  is parallel.

Theorem 1.3 does not hold in higher dimensions (cf. [17]). For  $m = -1$ , a two-dimensional metric in the form  $F = c\beta + \alpha^2/\beta$  is possibly Not locally projectively flat, although it is always Douglasian. For  $m = -3$ , by (5) and (6), we can construct two-dimensional metrics in the form  $F = c\beta + \alpha^4/\beta^3$  with  $c \neq 0$  which are Douglasian but Not locally projectively flat. See the examples in the last section. Besides, the local structure has been determined in [18] for  $F = c\beta + \alpha^2/\beta$ , or  $F = c\beta + \alpha^4/\beta^3$  which is locally projectively flat with constant flag curvature in dimension  $n \geq 2$ .

**Open Problem:** Determine the local structure of the two-dimensional metric  $F = c\beta + \alpha^2/\beta$ , or  $F = c\beta + \alpha^4/\beta^3$  ( $c \neq 0$ ) which is locally projectively flat.

## 2 Preliminaries

Let  $F = F(x, y)$  be a Finsler metric on an  $n$ -dimensional manifold  $M$ . In local coordinates, the spray coefficients  $G^i$  are defined by

$$G^i := \frac{1}{4}g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}. \quad (10)$$

If  $F$  is a Douglas metric, then  $G^i$  are in the following form:

$$G^i = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k + P(x, y)y^i, \quad (11)$$

where  $\Gamma_{jk}^i(x)$  are local functions on  $M$  and  $P(x, y)$  is a local positively homogeneous function of degree one in  $y$ . It is easy to see that  $F$  is a Douglas metric if and only if  $G^i y^j - G^j y^i$  is a homogeneous polynomial in  $(y^i)$  of degree three, which by (11) can be written as ([3]),

$$G^i y^j - G^j y^i = \frac{1}{2}(\Gamma_{kl}^i y^j - \Gamma_{kl}^j y^i) y^k y^l.$$

According to G. Hamel's result, a Finsler metric  $F$  is projectively flat in  $U$  if and only if

$$F_{x^m y^l} y^m - F_{x^l} = 0. \quad (12)$$

The above formula implies that  $G^i = P y^i$  with  $P$  given by

$$P = \frac{F_{x^m} y^m}{2F}. \quad (13)$$

Consider an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$ . The spray coefficients  $G_\alpha^i$  of  $\alpha$  are given by

$$G_\alpha^i = \frac{1}{4}a^{il}\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^l}\}.$$

Let  $\nabla\beta = b_{i|j}y^i dx^j$  denote the covariant derivatives of  $\beta$  with respect to  $\alpha$  and define

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad s^i := a^{ik} s_k,$$

where  $b^i := a^{ij} b_j$  and  $(a^{ij})$  is the inverse of  $(a_{ij})$ . By (10) again, the spray coefficients  $G^i$  of  $F$  are given by:

$$G^i = G_\alpha^i + \alpha Q s_0^i + \alpha^{-1} \Theta(-2\alpha Q s_0 + r_{00}) y^i + \Psi(-2\alpha Q s_0 + r_{00}) b^i, \quad (14)$$

where  $s_j^i = a^{ik} s_{kj}$ ,  $s_0^i = s_k^i y^k$ ,  $s_i = b^k s_{ki}$ ,  $s_0 = s_i y^i$ , and

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{Q - sQ'}{2\Delta}, \quad \Psi := \frac{Q'}{2\Delta}, \quad \Delta := 1 + sQ + (b^2 - s^2)Q'.$$

By (14) one can see that  $F = \alpha\phi(\beta/\alpha)$  is a Douglas metric if and only if

$$\alpha Q(s_0^i y^j - s_0^j y^i) + \Psi(-2\alpha Q s_0 + r_{00})(b^i y^j - b^j y^i) = \frac{1}{2}(G_{kl}^i y^j - G_{kl}^j y^i) y^k y^l, \quad (15)$$

where  $G_{kl}^i := \Gamma_{kl}^i - \gamma_{kl}^i$ ,  $\Gamma_{kl}^i$  are given in (11) and  $\gamma_{kl}^i := \partial^2 G_\alpha^i / \partial y^k \partial y^l$ .

Further,  $F = \alpha\phi(\beta/\alpha)$  is projectively flat on  $U \subset R^n$  if and only if

$$(a_{ml}\alpha^2 - y_m y_l) G_\alpha^m + \alpha^3 Q s_{l0} + \Psi\alpha(-2\alpha Q s_0 + r_{00})(\alpha b_l - s y_l) = 0, \quad (16)$$

where  $y_l = a_{ml} y^m$ .

### 3 Equations in a Special Coordinate System

In order to prove Theorems 4.1 and 5.1 below, one has to simplify (15) and (16). The main technique is to fix a point and choose a special coordinate system  $(s, y^a)$  as in [12] [13].

Fix an arbitrary point  $x \in M$  and take an orthogonal basis  $\{e_i\}$  at  $x$  such that

$$\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}, \quad \beta = by^1.$$

Then we change coordinates  $(y^i)$  to  $(s, y^a)$  such that

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha},$$

where  $\bar{\alpha} = \sqrt{\sum_{a=2}^n (y^a)^2}$ . Let

$$\bar{r}_{10} := r_{1a}y^a, \quad \bar{r}_{00} := r_{ab}y^ay^b, \quad \bar{s}_0 := s_ay^a.$$

We have  $\bar{s}_0 = b\bar{s}_{10}$ ,  $s_1 = bs_{11} = 0$ .

The following two lemmas are needed and are trivial:

**Lemma 3.1** *Under the special local coordinate system at  $x$ , if  $b = \text{constant}$ , then  $r_{12} + s_{12} = 0$  at  $x$ .*

**Lemma 3.2** *For  $n \geq 2$ , suppose  $p + q\bar{\alpha} = 0$ , where  $p = p(\bar{y})$  and  $q = q(\bar{y})$  are homogeneous polynomials in  $\bar{y} = (y^a)$ , then  $p = 0, q = 0$ .*

By the above coordinate  $(s, y^a)$  and using (15) and (16), it follows from [6] [12] we have the following results:

**Proposition 3.3** ( $n = 2$ ) *An  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  is a Douglas metric if and only if*

$$\frac{s^2}{2(b^2 - s^2)}(G_{11}^1 - G_{12}^2 - G_{21}^2) + \frac{1}{2}G_{22}^1 = b\Psi\left(\frac{r_{11}s^2}{b^2 - s^2} + r_{22}\right), \quad (17)$$

$$\left(-\frac{s^2}{b^2 - s^2} + 2\Psi b^2 - 1\right)bQs_{12} - 2b\Psi r_{12}s = \frac{G_{11}^2}{2(b^2 - s^2)}s^3 + \frac{1}{2}(G_{22}^2 - G_{12}^1 - G_{21}^1)s, \quad (18)$$

where  $G_{jk}^i$  are defined in (15).

**Proposition 3.4** ( $n = 2$ ) *An  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  is projectively flat if and only if*

$$\frac{s^2}{2(b^2 - s^2)}(-\tilde{G}_{11}^1 + 2\tilde{G}_{12}^2) - \frac{1}{2}\tilde{G}_{22}^1 = b\Psi\left(\frac{s^2}{b^2 - s^2}r_{11} + r_{22}\right), \quad (19)$$

$$\frac{1}{b^2 - s^2}[2\Psi(b^2 - s^2) - 1]b^3Qs_{12} - 2b\Psi r_{12}s = -\frac{\tilde{G}_{11}^2}{2(b^2 - s^2)}s^3 + \frac{1}{2}(-\tilde{G}_{22}^2 + 2\tilde{G}_{12}^1)s, \quad (20)$$

where  $\tilde{G}_{jk}^i := \frac{\partial^2 G_{jk}^i}{\partial y^j \partial y^k}$  are the connection coefficients of  $\alpha$ .

Comparing (17) and (19), (18) and (20), it is easy to see that if we can solve  $G_{jk}^i$  from (17) and (18), then we can solve  $\tilde{G}_{jk}^i$  from (19) and (20). In the following we only consider (17) and (18), from which we will solve  $G_{jk}^i$ .

## 4 Douglas $(\alpha, \beta)$ -metrics

In this section, we characterize a class of two-dimensional  $(\alpha, \beta)$ -metrics which are Douglasian. We have the following theorem.

**Theorem 4.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be a two-dimensional  $(\alpha, \beta)$ -metric on an open subset  $U \subset R^2$ , where  $\phi$  satisfies (2). Suppose that  $\beta$  is not parallel with respect to  $\alpha$ . Then  $F$  is a Douglas metric if and only if one of the following cases holds:*

(i)  $\phi$  is given by

$$\phi(s) = cs + \frac{1}{s}, \quad (21)$$

and  $\alpha$  and  $\beta$  are arbitrary, where  $c$  is a constant.

(ii)  $\phi$  and  $\beta$  satisfy

$$\phi(s) = k_1s + \frac{2k_2}{s} + \frac{1}{s^3}, \quad (22)$$

$$r_{ij} = -2\tau\{3b^2a_{ij} + (k_2b^2 - 2)b_ib_j\} + \frac{(3k_1 + k_2^2)b^4 - 4}{8b^2(1 + k_2b^2)}(b_is_j + b_js_i), \quad (23)$$

where  $\tau = \tau(x)$  is a scalar function and  $k_1, k_2$  are constant satisfying  $1 + k_2b^2 \neq 0$ .

(iii)  $\phi$  and  $\beta$  satisfy

$$\phi(s) = k_1s + s^m(1 + k_2s^2)^{\frac{1-m}{2}}, \quad (24)$$

$$b_{i|j} = 2\tau\{mb^2a_{ij} - (m + 1 + k_2b^2)b_ib_j\}, \quad (25)$$

where  $\tau = \tau(x)$  is a scalar function and  $k_1, k_2$  are constant.

(iv)  $\phi$  and  $\beta$  satisfy

$$\phi(s) = s^m(1 + ks^2)^{\frac{1-m}{2}}, \quad (26)$$

$$r_{ij} = 2\tau\{mb^2a_{ij} - (m + 1 + kb^2)b_ib_j\} - \frac{m + 1 + 2kb^2}{(m - 1)b^2}(b_is_j + b_js_i), \quad (27)$$

where  $k$  is constant and  $\tau = \tau(x)$  is a scalar.

(v)  $\phi$  and  $\beta$  satisfy

$$\phi(s) = mb^2\sqrt{b^2 - s^2} \int_0^s \frac{1}{(b^2 - t^2)^{3/2}} \left( \frac{t}{\sqrt{1 - kt^2}} \right)^{m-1} dt, \quad (28)$$

$$r_{ij} = -\frac{1}{b^2}(b_is_j + b_js_i), \quad (29)$$

where  $k$  is a constant.

In Theorem 4.1 (iv), if  $b = \text{constant}$ , then  $k = -1/b^2$  in (26)–(27), and we get

$$\phi(s) = s^m \left\{ 1 - \left( \frac{s}{b} \right)^2 \right\}^{\frac{1-m}{2}}, \quad (30)$$

$$r_{ij} = 2\bar{\tau}(b^2a_{ij} - b_ib_j) - \frac{1}{b^2}(b_is_j + b_js_i), \quad (31)$$

where  $\bar{\tau} := m\tau$ . Note that if  $n = 2$ , (31) is equivalent to  $b = \text{constant}$  (see [8]). When  $k_1 = k_2^2$ , Theorem 4.1 (ii) is a special case of Theorem 4.1 (iv).

#### 4.1 $(r_{11}, r_{22}) \neq (0, 0)$

**Step I:** We first consider the equation (17). Since  $(r_{11}, r_{22}) \neq (0, 0)$ , (17) can be written as

$$2\Psi = \frac{\lambda s^2 + \mu(b^2 - s^2)}{\delta s^2 + \eta(b^2 - s^2)}, \quad (32)$$

where  $\lambda = \lambda(x), \mu = \mu(x), \delta = \delta(x), \eta = \eta(x)$  are scalar functions. Since  $\phi$  satisfies (2),  $F$  is not of Randers type and we have  $\lambda\eta - \mu\delta \neq 0$ . Then by (17) and (32), for some scalar  $\bar{\tau} = \bar{\tau}(x)$ , there hold (see also [6])

$$r_{11} = 2b^2\delta\bar{\tau}, \quad r_{22} = 2b^2\eta\bar{\tau}, \quad (33)$$

$$G_{11}^1 = G_{12}^2 + G_{21}^2 + 2\lambda b^3\bar{\tau}, \quad G_{22}^1 = 2\mu b^3\bar{\tau}. \quad (34)$$

Rewrite (32) as follows

$$[\delta s^2 + \eta(b^2 - s^2)]\phi'' = [\lambda s^2 + \mu(b^2 - s^2)][\phi - s\phi' + (b^2 - s^2)\phi'']. \quad (35)$$

Plug

$$\phi(s) = cs + s^m(1 + a_{m+1}s + a_{m+2}s^2 + a_{m+3}s^3 + a_{m+4}s^4 + a_{m+5}s^5) + o(s^{m+5}) \quad (36)$$

into (35). Let  $p_i$  be the coefficients of  $s^i$  in (35). First  $p_{m-2} = 0$  gives

$$\eta = \mu b^2. \quad (37)$$

Plugging (37) into  $p_m = 0$  yields

$$\delta = \lambda b^2 - \frac{m+1}{m}\mu b^2. \quad (38)$$

**Case A.** Assume  $m = -1$ . Plug (37), (38) and  $m = -1$  into (35) and then we get

$$s^2\phi'' + s\phi' - \phi = 0,$$

whose solution is given by (21). By (33), (34), (37) and (38) we obtain

$$G_{11}^1 = G_{12}^2 + G_{21}^2 + \frac{r_{11}}{b}, \quad G_{22}^1 = \frac{r_{22}}{b}. \quad (39)$$

**Case B.** Assume  $m \neq -1$ . Plugging (37) and (38) into  $p_{m+2} = 0$  yields

$$\lambda = [m(m-1) + 2a_{m+2}b^2]\epsilon, \quad \mu = m(m-1)\epsilon, \quad (40)$$

where  $\epsilon = \epsilon(x) \neq 0$  is a scalar. It is easy to see that

$$\lambda\eta - \mu\delta = m(m+1)(m-1)^2b^2\epsilon^2 \neq 0. \quad (41)$$

Plug (37), (38) and (40) into (32) and we get

$$2\Psi = \frac{\phi''}{\phi - s\phi' + (b^2 - s^2)\phi''} = \frac{m(m-1) + 2a_{m+2}s^2}{m(m-1)b^2 + (1 - m^2 + 2a_{m+2}b^2)s^2}. \quad (42)$$

**Step II:** Next we solve the equation (18). Put  $\xi := G_{12}^1 + G_{21}^1 - G_{22}^2$ .

**Case A.** Assume  $m = -1$ . Plug (21) into (18) and we have

$$\xi = \frac{2r_{12} - s_{12}}{b}, \quad G_{11}^2 = \frac{1 - cb^2}{b}s_{12}. \quad (43)$$

Now we have seen that a two dimensional metric  $F = k\beta + \alpha^2/\beta$  is always Douglasian.

**Case B.** Assume  $m \neq -1$ . Plug (42) into (18) and we have

$$\begin{aligned} 0 = & -2b(b^2 - s^2)[m(m-1) + 2a_{m+2}s^2]r_{12}(\phi - s\phi') - 2b^3\phi's(1 - m + 2a_{m+2}s^2)s_{12} \\ & + [(1 - m^2 + 2a_{m+2}b^2)s^2 + m(m-1)b^2][(b^2 - s^2)\xi - G_{11}^2s^2](\phi - s\phi'). \end{aligned} \quad (44)$$

Plug the expansion as in (36) into (44). Let  $p_i$  denote the coefficient of  $s^i$  in (44). For convenience, we put

$$a_{m+2} = \frac{1}{2}c_1, \quad a_{m+4} = \frac{1}{8} \frac{(m^2 + 5m + 4)c_1^2 - 2c_2}{m(m-1)}. \quad (45)$$

Note that in the next computation, when  $c \neq 0$  and  $m = 3$ ,  $p_m$  has different result from that for  $m \neq 3$ , but there is no effect on the final result. So we only consider  $m \neq 3$  in the computation for  $p_m$ . Solving the system  $p_m = 0, p_{m+2} = 0$  and  $p_{m+4} = 0$  yields the following two cases:

(i) If

$$m - 1 - c_1b^2 \neq 0, \quad (46)$$

then we have

$$r_{12} = \frac{c_2b^4 - (m^2 - 1)(m + 3)c_1b^2 + (m^2 - 1)^2}{(m + 1)(m - 1)^2(1 - m + c_1b^2)}s_{12}, \quad (47)$$

$$G_{11}^2 = 2 \frac{\{(m + 2)c_1^2 - c_2\}b^4 + m(m^2 - 1)c_1b^2 - m^2(m - 1)^2}{m(m - 1)^2b(1 - m + c_1b^2)}s_{12}, \quad (48)$$

$$\xi = 2 \frac{c_2b^4 - (m^2 - 1)(m + 2)c_1b^2 + m(m + 1)(m - 1)^2}{(m + 1)(m - 1)^2b(1 - m + c_1b^2)}s_{12}. \quad (49)$$

(ii) If

$$m - 1 - c_1b^2 = 0, \quad (50)$$

then we get

(iia) If  $s_{12} = 0$ , then  $r_{12} = 0$  by Lemma 3.1 since  $b = \text{constant}$ .

(iib) If  $s_{12} \neq 0$ , then

$$c_1 = \frac{m - 1}{b^2}, \quad c_2 = 2(m + 1)c_1^2. \quad (51)$$

Now we can determine  $Q$  under  $s_{12} \neq 0$ , and  $\phi$  under two cases:  $s_{12} = 0$  and  $s_{12} \neq 0$ .

**Case B1:** Assume  $s_{12} = 0$ .

If (46) holds, then  $r_{12} = 0$  by (47). If (50) holds, we also get  $r_{12} = 0$ . Then by (33), (37), (38) and (40) we get the expression of  $b_{i|j}$  given by (25), where we put

$$\tau := (m - 1)\epsilon b^2 \bar{\tau}. \quad (52)$$



By (42) we get

$$\phi'' = \frac{-m + k_2 s^2}{(1 + k_2 s^2)s^2}(\phi - s\phi'), \quad (53)$$

where we put

$$k_1 = a_1, \quad k_2 = -2a_{m+2}/(m-1).$$

Solving the differential equation (53) gives (24). This class belongs to Theorem 4.1(iii).

**Case B2:** Assume  $s_{12} \neq 0$ .

**B2(i):** Suppose (46). We plug (47), (48) and (49) into (44), and then we obtain

$$Q = \frac{[(m+2)c_1^2 - c_2]s^4 + m(m^2 - 1)c_1 s^2 - m^2(m-1)^2}{m(m-1)^2 s(1 - m + c_1 s^2)}, \quad (54)$$

By (54) we can get  $\phi'$  and by differentiating it we get  $\phi''$ . Then plugging  $\phi'$  and  $\phi''$  into (42) we have the following two cases:

**B2(i)(1):**  $m = -3$ . In this case, (54) implies (42). Then solving (54) gives (22), where we define  $k_1, k_2$  by

$$k_1 := -\frac{c_2 + c_1^2}{48}, \quad k_2 := \frac{c_1}{4}. \quad (55)$$

By (33), (37), (38), (40) and (47) we obtain (23), where  $\tau = \tau(x)$  is defined by (52) with  $m = -3$ . Then we get Theorem 4.1(ii).

**B2(i)(2):**  $m \neq -3$ . In this case, we have

$$c_2 = 2(m+1)c_1^2. \quad (56)$$

Plug (56) into (47) and then we have

$$r_{12} = -\frac{m+1+2kb^2}{m-1}s_{12}, \quad (57)$$

where  $k = -c_1/(m-1)$ . By (33), (37), (38), (40) and (57) we obtain (27) with  $\tau := (m-1)b^2\epsilon\bar{\tau}$ . Plugging (56) and  $c_1 = (1-m)k$  into (54) yields

$$\frac{\phi'}{\phi - s\phi'} = -\frac{m + ks^2}{(m-1)s}.$$

Thus we easily get  $\phi$  given by (26). This class belongs to Theorem 4.1(iv).

**B2(ii):** Suppose (50). Then  $r_{12} = -s_{12}$ . Plug (51) into (37), (38) and (40), then we get  $\delta, \eta$ . Plug  $\delta, \eta$  into (33), then we get  $r_{11}, r_{22}$ . Plus  $r_{12} = -s_{12}$  we obtain (31) for some scalar  $\bar{\tau} = \bar{\tau}(x)$ . Similarly as above we get  $\phi$  given by (30). This class belongs to Theorem 4.1(iv).

## 4.2 $(r_{11}, r_{22}) = (0, 0)$

Since  $\beta$  is not parallel and  $(r_{11}, r_{22}) = (0, 0)$ , we will see that  $s_{12} \neq 0$  from the following proof for different cases. It follows from (17) that

$$G_{22}^1 = 0, \quad G_{11}^1 = G_{12}^2 + G_{21}^2.$$

Plugging the expressions of  $Q$  and  $\Psi$  into (18) yields

$$\begin{aligned} & s(b^2 - s^2)[2br_{12} + G_{11}^2 s^2 - (b^2 - s^2)\xi]\phi'' \\ & + s[G_{11}^2 s^2 - (b^2 - s^2)\xi](\phi - s\phi') + 2b^3 s_{12}\phi' = 0, \end{aligned} \quad (58)$$

where  $\xi := G_{12}^1 + G_{21}^1 - G_{22}^2$ .

Plug

$$\phi = a_1 s + s^m (1 + a_{m+1} s + a_{m+2} s^2 + a_{m+3} s^3 + a_{m+4} s^4 + \dots)$$

into (58) and let  $p_i$  be the coefficient of  $s^i$  in (58). All  $p_i$ 's are zero.

By  $p_{m-1} = 0$  we have

$$r_{12} = \frac{1}{2} b \xi - \frac{1}{m-1} s_{12}. \quad (59)$$

Plugging (59) into  $p_{m+1} = 0$  yields

$$G_{11}^2 = -\frac{m+1}{m} \xi + \frac{2}{m-1} \left\{ \frac{2(m+2)ba_{m+2}}{m(m-1)} - \frac{1}{b} \right\} s_{12}. \quad (60)$$

**Case I:** Assume  $b \neq \text{constant}$ . We will get Theorem 4.1(i), (ii) and (iv) in a special case.

**Case IA:**  $m = -1$ . By the discussion of the two cases (1) and (2) below we get  $\phi(s)$  given by (21) for some constant  $c$ .

(1). Assume  $a_{2k} \neq 0$  for some minimal integer  $k \geq 0$ . Plugging  $a_{2k-2} = 0$  into  $p_{2k+1} = 0$  gives

$$\xi = \left\{ \frac{2ka_1}{1+2k} - \frac{2(1+k)(3+2k)a_{2k+2}}{(2k+1)(2k-1)a_{2k}} \right\} b s_{12}. \quad (61)$$

Then substitute (59), (60) and (61) into (58), and using  $b \neq \text{constant}$  we obtain  $\phi(s)$  given by (21) for some constant  $c$ . Thus  $a_{2k} = 0$  for all integers  $k \geq 0$  by (21). This contradicts with our assumption.

(2). Assume  $a_{2k+1} \neq 0$  for some minimal integer  $k \geq 1$ . If  $k = 1$ , we get  $s_{12} = 0$  by  $p_2 = 0$ . If  $k = 2$ , we get  $s_{12} = 0$  by plugging  $a_3 = 0$  into  $p_4 = 0$ . If  $k \geq 3$ , we get  $s_{12} = 0$  by plugging  $a_{2k-3} = 0$  and  $a_{2k-1} = 0$  into  $p_{2k} = 0$ . Now substitute (59), (60) and  $s_{12} = 0$  into (58), and then we get

$$\xi s(b^2 - s^2)(s^2 \phi'' + s \phi' - \phi) = 0. \quad (62)$$

If  $\xi = 0$ , then we have  $r_{12} = 0$  by (59) and  $s_{12} = 0$ . Thus  $\xi \neq 0$ . Then by (62) we obtain  $\phi(s)$  given by (21) for some constant  $c$ . So  $a_{2k+1} = 0$  for all integers  $k \geq 1$  by (21). This again contradicts with our assumption.

**Case IB:**  $m \neq -1$ . By aid of (45), plugging (59) and (60) into  $p_{m+3} = 0$  yields

$$\xi = \frac{2 \left\{ [(m+4)c_2 - 2(m+1)(m+3)c_1^2] b^4 - (m^2 - 1)[(m+2)c_1 b^2 - m(m-1)] \right\}}{(m+1)(m-1)^2 b(1-m+c_1 b^2)} s_{12}. \quad (63)$$

By (59) and (63) we conclude that if  $s_{12} = 0$ , then we have  $r_{12} = 0$ . So in this case, we have  $s_{12} \neq 0$ . Plug (59), (60) and (63) into (58) and then we obtain an equation in the form

$$f_0 + f_2 b^2 + f_4 b^4 = 0,$$

where  $f_0, f_2$  and  $f_4$  are ODEs about  $\phi$ . Since  $b \neq \text{constant}$ , we have  $f_0 = 0, f_2 = 0, f_4 = 0$ . This system is equivalent to  $f_0 = 0, f_2 = 0$ .

(1).  $m = -3$ . In this case, solving the system  $f_0 = 0, f_2 = 0$  gives (22) for some constants  $k_1, k_2$ . Meanwhile we get (23) with  $\tau = 0$  by (59), (63),  $r_{11} = 0$  and  $r_{22} = 0$ .

(2).  $m = -4$ . We get  $\phi$  given by (26) with  $m = -4$ .

(3).  $m \neq -3, -4$ . Solving the system  $f_0 = 0, f_2 = 0$  we can first show that

$$c_2 = 2(m+1)c_1^2. \quad (64)$$

Plugging (64) into the system  $f_0 = 0, f_2 = 0$  again we get the solution of  $\phi$  given by (26) with  $k = -c_1/(m-1)$ .

If  $m \neq -3$ , plug (64) into (59), (60) and (63) and we have

$$r_{12} = \frac{1+m+2kb^2}{1-m}s_{12}, \quad G_{11}^2 = \frac{2(m+kb^2)}{(m-1)b}s_{12}, \quad \xi = \frac{2(m+2kb^2)}{(1-m)b}s_{12}. \quad (65)$$

Now by the expression of  $r_{12}$  in (65) we get (27) with  $\tau = 0$ .

**Case II:** Assume  $b = \text{constant}$ . We will show this case gives the class Theorem 4.1(v).

Since  $r_{12} + s_{12} = 0$ , it follows from (59) that

$$\xi = \frac{2(m-2)}{(1-m)b}s_{12}. \quad (66)$$

If  $m = -2$ , substituting (66) and (60) into (58) yields (68) with  $m = -2$  and  $k = 1/b^2$ .

If  $m \neq -2$ , plug (66) into (60) and we have

$$G_{11}^2 = \frac{2(m-2+kb^2)}{(m-1)b}s_{12}, \quad (67)$$

where we put

$$a_{m+2} = \frac{(m-1)(2+mb^2)}{2(m+2)b^2}.$$

Now plugging  $r_{12} = -s_{12}$ , (66) and (67) into (58) yields

$$\frac{\phi - s\phi' + (b^2 - s^2)\phi''}{s\phi + (b^2 - s^2)\phi'} = \frac{m-1}{s(1-ks^2)}. \quad (68)$$

Let

$$\Phi := s\phi(s) + (b^2 - s^2)\phi'(s).$$

Then (68) becomes

$$\frac{\Phi'}{\Phi} = \frac{m-1}{s(1-ks^2)}.$$

We get

$$\Phi = c \left( \frac{s}{\sqrt{1-ks^2}} \right)^{m-1},$$

where  $c$  is a constant. Then we can easily get  $\phi$  given by (28). And (29) follows from  $r_{11} = 0, r_{22} = 0$  and  $r_{12} = -s_{12}$ .

## 5 Projectively flat $(\alpha, \beta)$ -metrics

In this section, we characterize a two-dimensional  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  satisfying (2) which is projectively flat. We have the following theorem.

**Theorem 5.1** *Let  $F = \alpha\phi(s)$ ,  $s = \beta/\alpha$ , be an  $(\alpha, \beta)$ -metric on an open subset  $U \subset R^2$ . Suppose that  $\beta$  is not parallel with respect to  $\alpha$  and  $\phi$  satisfies (2). Let  $G_\alpha^i$  be the spray coefficients of  $\alpha$ . Then  $F$  is projectively flat in  $U$  with  $G^i = P(x, y)y^i$  if and only if one of the following cases holds:*

(i)  $\phi(s)$  satisfy (21), and  $G_\alpha^i$  satisfy

$$G_\alpha^i = \rho y^i - \frac{r_{00}}{2b^2} b^i - \frac{\alpha^2 - c\beta^2}{2b^2} s^i. \quad (69)$$

In this case, the projective factor  $P$  is given by

$$P = \rho - \frac{1}{b^2(\alpha^2 + c\beta^2)} \left\{ (\alpha^2 - c\beta^2)s_0 + r_{00}\beta \right\}. \quad (70)$$

(ii)  $\phi(s)$  and  $\beta$  satisfy (22) and (23), and  $G_\alpha^i$  satisfy

$$G_\alpha^i = \rho y^i + \tau(3\alpha^2 + k_2\beta^2)b^i + \left\{ \frac{k_1 - k_2^2}{8(1 + k_2b^2)}(3b^2\alpha^2 - \beta^2) + \left(\frac{k_2}{2} - \frac{3}{4b^2}\right)\alpha^2 - \frac{k_2}{b^2}\beta^2 \right\} s^i. \quad (71)$$

In this case, the projective factor  $P$  is given by

$$\begin{aligned} P &= \rho + 2\tau\beta \left\{ 3 - \frac{2c\beta^4}{\alpha^4 + c\beta^4 + k_2\beta^2(2\alpha^2 + k_2\beta^2)} \right\} + \left( \frac{k_2b^2 - 3}{2b^2} + T \right) s_0, \\ T &= c \frac{4\beta^2(2\beta^2 - b^2\alpha^2) + 3b^4(\alpha^4 + c\beta^4) + k_2b^2\beta^2(6b^2\alpha^2 + 4\beta^2 + 3k_2b^2\beta^2)}{8b^2(1 + k_2b^2)[\alpha^4 + c\beta^4 + k_2\beta^2(2\alpha^2 + k_2\beta^2)]}, \\ c &= k_1 - k_2^2. \end{aligned} \quad (72)$$

(iii)  $\phi(s)$  and  $\beta$  satisfy (24) and (25), and  $G_\alpha^i$  satisfy

$$G_\alpha^i = \rho y^i - \tau(m\alpha^2 - k_2\beta^2)b^i. \quad (73)$$

In this case, the projective factor  $P$  is given by

$$P = \rho + \tau\alpha \left\{ s(-m + k_2s^2) - s^2(1 + k_2s^2) \frac{\phi'}{\phi} \right\}. \quad (74)$$

(iv)  $\phi(s)$  and  $\beta$  satisfy (26) and (27), and  $G_\alpha^i$  satisfy

$$G_\alpha^i = \rho y^i - \tau(m\alpha^2 - k\beta^2)b^i + \frac{1}{1-m} \left\{ \left(2k + \frac{m}{b^2}\right)\alpha^2 - \frac{k}{b^2}\beta^2 \right\} s^i. \quad (75)$$

In this case, the projective factor  $P$  is given by

$$P = \rho - 2m\tau\beta - \frac{2(m + kb^2)}{(m-1)b^2} s_0. \quad (76)$$

(v)  $\phi(s)$  and  $\beta$  satisfy (28) and (29), and  $G_\alpha^i$  satisfy

$$G_\alpha^i = \rho y^i - \frac{(m-2)\alpha^2 + k\beta^2}{(m-1)b^2} s^i. \quad (77)$$

In this case, the projective factor  $P$  is given by

$$P = \rho + \frac{1}{(m-1)b^2} \left\{ s(ks^2 - 1) \frac{\phi'}{\phi} - ks^2 - m + 2 \right\} s_0. \quad (78)$$

The above function  $\rho = \rho_i(x)y^i$  is a 1-form.

To complete the proof of Theorem 5.1, we only need to solve  $\tilde{G}_{jk}^i$  from (19) and (20), and all projective factors for every class in Theorem 4.1 when  $F$  is projectively flat.

**Remark 5.2** In Theorem 5.1, when  $\beta$  is not closed, since  $n = 2$ , we can express  $G^i$  and  $P$  for every class in different forms with different choices of  $\rho$  using  $s_{12}$  in the following proof. However, we can verify conversely that these different forms are equivalent to one another using the dimension  $n = 2$ .

## 5.1 The Spray Coefficients of $\alpha$

In this subsection we will show the expressions of the spray coefficients  $G_\alpha^i$  for each class in Theorem 4.1 when  $F$  is projectively flat. Note that by  $\tilde{G}_{jk}^i = \frac{\partial^2 G_\alpha^i}{\partial y^j \partial y^k}$ , the spray  $G_\alpha^i$  of  $\alpha$  can be expressed as

$$G_\alpha^i = \frac{1}{2} \tilde{G}_{jk}^i y^j y^k.$$

**Case I:** Suppose  $(r_{11}, r_{22}) \neq (0, 0)$ .

(1). Assume  $m = -1$ . It follows from (39) and (43) that  $G_\alpha^i$  are given by (69), where  $\rho$  is defined by

$$\rho := \tilde{G}_{12}^2 y^1 + (\tilde{G}_{12}^1 + \frac{r_{12}}{b}) y^2.$$

(2). Assume  $m \neq -1$ . By (34) we have

$$\tilde{G}_{11}^1 = 2t_1 - 2\lambda b^3 \bar{\tau}, \quad \tilde{G}_{22}^1 = -2\mu b^3 \bar{\tau}, \quad \tilde{G}_{12}^1 = t_2, \quad \tilde{G}_{12}^2 = t_1. \quad (79)$$

By (48) and (49) we get

$$\tilde{G}_{11}^2 = -2 \frac{\{(m+2)c_1^2 - c_2\}b^4 + m(m^2-1)c_1b^2 - m^2(m-1)^2}{m(m-1)^2b(1-m+c_1b^2)} s_{12}, \quad (80)$$

$$\tilde{G}_{22}^2 = 2\tilde{G}_{12}^1 + 2 \frac{c_2b^4 - (m^2-1)(m+2)c_1b^2 + m(m+1)(m-1)^2}{(m+1)(m-1)^2b(1-m+c_1b^2)} s_{12}. \quad (81)$$

Let  $\rho = t_i y^i$  and  $\tau$  be given by (52). If  $\beta$  is closed ( $s_{12} = 0$ ), then it follows from (37), (38) and (40) and (79)–(81) that (73) holds. If  $\beta$  is not closed, then if  $m = -3$  we get (71) from (79)–(81), where  $k_1, k_2$  are defined by (55); if  $m \neq -3$  we get (75) from (56), (79)–(81), where  $k = c_1/(1-m)$ ,

**Case II:** Suppose  $(r_{11}, r_{22}) = (0, 0)$ . Then by (19) we get

$$\tilde{G}_{22}^1 = 0, \quad \tilde{G}_{11}^1 = 2\tilde{G}_{12}^2 = 2t_1, \quad \tilde{G}_{12}^2 = t_1, \quad \tilde{G}_{12}^1 = t_2. \quad (82)$$

(1).  $b \neq \text{constant}$ . If  $m = -1$ , (69) has been proved. If  $m = -3$ , we get (71) with  $\tau = 0$ . If  $m \neq -1, -3$ , then it follows from (65) that

$$\tilde{G}_{11}^2 = -\frac{2(m+kb^2)}{(m-1)b} s_{12}, \quad \tilde{G}_{22}^2 = 2\tilde{G}_{12}^1 + \frac{2(m+2kb^2)}{(1-m)b} s_{12}. \quad (83)$$

Then by (82) and (83) we obtain (75) with  $\tau = 0$ .

(2).  $b = \text{constant}$ . It follows from (66) and (67) that

$$\tilde{G}_{11}^2 = -\frac{2(m-2+kb^2)}{(m-1)b} s_{12}, \quad \tilde{G}_{22}^2 = 2\tilde{G}_{12}^1 + \frac{2(m-2)}{(1-m)b} s_{12}. \quad (84)$$

Then by (82) and (84) we obtain (77).

## 5.2 The Projective Factors

In this subsection, we are going to show the projective factors for each class in Theorem 5.1.

(1): We first prove (74). By (25) we have

$$r_{00} = 2\tau\{mb^2\alpha^2 - (1 + m + k_2b^2)\beta^2\}. \quad (85)$$

Now plug  $s_{i0} = 0, s_0 = 0$  and (73), (53) and (85) into (14), and then we obtain (74).

(2): For the proofs to (76) and (78), since  $\beta$  may not be closed, it is not easy to show the projective factors in the initial *local projective coordinate system* (in such a coordinate system, geodesics are straight lines). However, it is easy to be solved by choosing another local projective coordinate system, and then returning to the the initial local projective coordinate system.

Fix an arbitrary point  $x_o \in U \subset R^2$ . By the above idea and a suitable affine transformation, we may assume  $(U, x^i)$  is a local projective coordinate system satisfying that  $\alpha_{x_o} = \sqrt{(y^1)^2 + (y^2)^2}$  and  $\beta_{x_o} = by^1$ . Then at  $x_o$  we have

$$s^1 = s_1 = 0, \quad s^2 = s_2 = bs_{12}, \quad s_0 = bs_{12}y^2, \quad b^1 = b_1 = b, \quad b^2 = b_2 = 0.$$

Suppose (26) and (27). Then it is easy to get

$$\begin{aligned} r_{00} &= 2\tau\{mb^2\alpha^2 - (1 + m + kb^2)\beta^2\} - \frac{2(1 + m + 2kb^2)}{(m-1)b^2}\beta s_0, \\ s_0^i &= \frac{1}{b^2}(s_0b^i - \beta s^i), \quad Q = \frac{m + ks^2}{(1-m)s}, \\ \Theta &= \frac{ms}{(1 + m + kb^2)s^2 - mb^2}, \quad \Psi = \frac{ks^2 - m}{2(1 + m + kb^2)s^2 - 2mb^2}. \end{aligned}$$

Plug (75) and all the above expressions into (14), and then at  $x_o$  we see that  $G^i = Py^i$ , where  $P$  is given by

$$P = \rho + 2m\tau by^1 - \frac{2(m + kb^2)}{(m-1)b}s_{12}y^2, \quad (86)$$

By using

$$bs_{12}y^2 = s_0, \quad by^1 = \beta,$$

it is easy to transform (86) into (76). Since  $x_o$  is arbitrarily chosen, (76) holds everywhere.

Suppose (28) and (29). Then it is easy to get (where we use (68) in place of (28))

$$\begin{aligned} s_0^i &= \frac{1}{b^2}(s_0b^i - \beta s^i), \quad r_{00} = -\frac{2}{b^2}\beta s_0, \\ \Theta &= \frac{s}{2(m-1)(b^2 - s^2)}\left\{s(ks^2 - 1)\frac{\phi'}{\phi} + 2 - m - ks^2\right\}, \\ \Psi &= \frac{\{ks^4 + (m-2)s^2 - (m-1)b^2\}\phi' + s(2 - m - ks^2)\phi}{2(1-m)(b^2 - s^2)\{(b^2 - s^2)\phi' + s\phi\}}. \end{aligned}$$

Plug (77) and all the above expressions into (14), and then by using the relations

$$bs_{12}y^2 = s_0, \quad (y^1)^2 + (y^2)^2 = \alpha^2, \quad by^1 = \beta, \quad \frac{\beta}{\alpha} = s,$$

we similarly get  $G^i = Py^i$ , where  $P$  is given by (78). The details are omitted.

Similarly we can get the projective factors of the other classes in Theorem 5.1. We omit the details.

## 6 Proof of Theorem 1.2

By Theorem 4.1, we can easily characterize a two-dimensional metric  $F = c\beta + \beta^m\alpha^{1-m}$  which is Douglasian as follows.

**Corollary 6.1** *Let  $F = c\beta + \beta^m\alpha^{1-m}$  be a two-dimensional  $(\alpha, \beta)$ -metric on an open subset  $U \subset \mathbb{R}^2$ , where  $c, m$  are constant with  $m \neq 0, 1$ . Then for some scalar function  $\tau = \tau(x)$ , we have the following cases:*

- (i) ( $m = -1$ )  $F$  is always a Douglas metric.
- (ii) ( $m = -3$ )  $F$  is a Douglas metric if and only if  $\beta$  satisfies

$$r_{ij} = 2\tau(-3b^2a_{ij} + 2b_ib_j) + \frac{3cb^4 - 4}{8b^2}(b_is_j + b_js_i). \quad (87)$$

- (iii) ( $c \neq 0; m \neq -1, -3$ )  $F$  is a Douglas metric if and only if  $\beta$  satisfies

$$b_{i|j} = 2\tau\{mb^2a_{ij} - (m+1)b_ib_j\}, \quad (88)$$

- (iv) ( $c = 0; m \neq -1$ )  $F$  is a Douglas metric if and only if  $\beta$  satisfies

$$r_{ij} = 2\tau\{mb^2a_{ij} - (m+1)b_ib_j\} - \frac{m+1}{(m-1)b^2}(b_is_j + b_js_i), \quad (89)$$

*Proof of Theorem 1.2 :*

**Case I:** Assume  $m = -3$ . In this case,  $F = c\beta + \alpha^4/\beta^3$ . Define a new Riemann metric  $\tilde{\alpha}$  and a 1-form  $\tilde{\beta}$  by

$$\tilde{\alpha} := \sqrt{\xi\alpha^2 + \eta\beta^2}, \quad \tilde{\beta} := \beta, \quad (90)$$

where

$$\xi := \frac{1}{b^2(4 + 3cb^4)}, \quad \eta := \frac{3(5 + 8cb^4 + 3c^2b^8)}{b^4(4 + 3cb^4)}\}.$$

Since  $F = c\beta + \alpha^4/\beta^3$  is a Douglas metric, we have (87). Now by (90) and (87), a direct computation gives

$$\tilde{r}_{ij} = -\frac{16\tau b^4}{(4 + 3cb^4)^2}\tilde{\alpha}_{ij}. \quad (91)$$

So  $\tilde{\beta} = \beta$  is a conformal form with respect to  $\tilde{\alpha}$ . Since  $\tilde{\alpha}$  is a two-dimensional Riemann metric, we can express  $\tilde{\alpha}$  locally as

$$\tilde{\alpha} := e^\sigma \sqrt{(y^1)^2 + (y^2)^2}, \quad (92)$$

where  $\sigma = \sigma(x)$  is a scalar function. We can obtain the local expression of  $\tilde{\beta} = \beta$  by (91) and (92) (see [20]). Then by the result in [20], we have

$$\tilde{\beta} = \tilde{b}_1y^1 + \tilde{b}_2y^2 = e^{2\sigma}(uy^1 + vy^2), \quad (93)$$

where  $u = u(x), v = v(x)$  are a pair of scalar functions such that

$$f(z) = u + iv, \quad z = x^1 + ix^2$$

is a complex analytic function. Finally, we give the relation between  $b^2 = \|\beta\|_\alpha^2$  with the triple  $(\sigma, u, v)$ , which can be done by computing the quantity  $\|\beta\|_\alpha^2$ . First, by (92) and (93) we get

$$\|\beta\|_\alpha^2 = e^{2\sigma}(u^2 + v^2). \quad (94)$$

On the other hand, by the definition of  $\tilde{\alpha}$  in (90), the inverse  $\tilde{\alpha}^{ij}$  of  $\tilde{\alpha}_{ij}$  is given by

$$\tilde{\alpha}^{ij} = \frac{1}{\xi} \left( a^{ij} - \frac{\eta b^i b^j}{\xi + \eta b^2} \right).$$

Now plug  $\xi$  and  $\eta$  into the above, and we obtain

$$\|\beta\|_\alpha^2 = \tilde{\alpha}^{ij} b_i b_j = \frac{b^4}{4 + 3cb^4}. \quad (95)$$

Thus by (94) and (95) we have

$$e^{2\sigma} = \frac{b^4}{(4 + 3cb^4)(u^2 + v^2)}. \quad (96)$$

Plug (96) into (92), (93) and (90) and then we get  $\alpha$  and  $\beta$  given by (5) and (6), where we define  $B := b^2$ .

**Case II:** Assume  $m \neq -1$  and  $c = 0$ . To prove this case, we first show the following lemma.

**Lemma 6.2** *Let  $\alpha$  be a two-dimensional Riemann metric on a manifold  $M$ . If there is a non-zero 1-form on  $M$  which is parallel with respect  $\alpha$ , then  $\alpha$  is flat.*

*Proof :* Let  $\beta$  be parallel 1-form with respect  $\alpha$ . Express  $\alpha$  and  $\beta$  locally as

$$\alpha = e^{\sigma(x)} \sqrt{(y^1)^2 + (y^2)^2}, \quad \beta = e^{2\sigma}(u(x)y^1 + v(x)y^2).$$

Since  $\beta$  is also a conformal form with respect to  $\alpha$ , by the result in [20] we know that  $u, v$  are a pair of conjugate harmonious functions, or equivalently,  $u, v$  satisfy

$$u_1 = v_2, \quad u_2 = -v_1, \quad (u_i := u_{x^i}, \quad v_i := v_{x^i}). \quad (97)$$

Put  $\|\beta\|_\alpha = 1$ . Then we have

$$\|\beta\|_\alpha^2 = e^{2\sigma}(u^2 + v^2) = 1.$$

So  $\alpha$  can be written as

$$\alpha = \sqrt{\frac{(y^1)^2 + (y^2)^2}{u^2 + v^2}}.$$

Now using (97), it can be shown that  $\alpha$  is of zero sectional curvature.

Q.E.D.

Since  $c = 0$ , we have  $F = \beta^m \alpha^{1-m}$ . Define a new Riemann metric  $\tilde{\alpha}$  and a 1-form  $\tilde{\beta}$  by (4). Since  $F = \beta^m \alpha^{1-m}$  is a Douglas metric ( $m \neq -1$ ), we have (89). By (4) and (89), a direct computation gives  $\tilde{r}_{ij} = 0$ . We can also give another simple proof. Since  $F$  is a Douglas metric and  $F$  keeps formally unchanged under (4), by (89) and using  $\tilde{b} = 1$  we have

$$\tilde{r}_{ij} = 2\tilde{\tau} \{ m\tilde{a}_{ij} - (m+1)\tilde{b}_i \tilde{b}_j \} - \frac{m+1}{m-1} (\tilde{b}_i \tilde{s}_j + \tilde{b}_j \tilde{s}_i). \quad (98)$$



Contracting (98) by  $\tilde{b}^i$  and then by  $\tilde{b}^j$  and using  $\tilde{r}_i + \tilde{s}_i = 0$ , it is easy to get  $\tilde{r}_{ij} = 0$ . In case of dimension  $n = 2$ , given any pair  $\alpha$  and  $\beta$ , we always have

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i). \quad (99)$$

So by  $\tilde{s}_j = 0$  and (99) we have  $\tilde{s}_{ij} = 0$ , which imply that  $\tilde{\beta}$  is closed. Thus  $\tilde{\beta}$  is parallel with respect to  $\tilde{\alpha}$ . Therefore, by Lemma 6.2,  $\tilde{\alpha}$  is flat. Thus  $\tilde{\alpha}$  and  $\tilde{\beta}$  can be locally written in the form (8).

**Case III:** Assume  $m \neq -1, c \neq 0, m \neq -3$ . Since  $F = c\beta + \beta^m \alpha^{1-m}$  is a Douglas metric, we get (88). Under the deformation (4), (88) becomes  $\tilde{b}_{i|j} = 0$ . So by Lemma 6.2, we again obtain (8). By (9) and the fact that  $\beta = \eta\tilde{\beta}$  is closed, we get  $\eta = \eta(x^1)$ . Q.E.D.

**Remark 6.3** We can give another useful local representation corresponding to Theorem 1.2(iii) and (iv). Define

$$\tilde{\alpha} := \sqrt{\frac{(y^1)^2 + (y^2)^2}{u^2 + v^2}}, \quad \tilde{\beta} := \frac{uy^1 + vy^2}{u^2 + v^2}, \quad (100)$$

where  $u = u(x), v = v(x)$  satisfy (97). Then  $\tilde{\alpha}$  is flat and  $\tilde{\beta}$  is parallel with respect to  $\tilde{\alpha}$ . Put  $\alpha$  and  $\beta$  as that in (9). Then  $F = \beta^m \alpha^{1-m}$  with  $m \neq 0, 1$  is locally Minkowskian. If  $\eta, u, v$  satisfy

$$u_1 = v_2, \quad u_2 = -v_1, \quad \eta_1 v = \eta_2 u, \quad (u_i := u_{x^i}, \text{etc.}), \quad (101)$$

then  $\beta$  is closed, and  $F = c\beta + \beta^m \alpha^{1-m}$  with  $m \neq 0, 1$  is locally projectively flat by the proof to Theorem 1.3 below. However, there is no the relation  $G^i = Py^i$  in such a coordinate system.

## 7 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 and thus the local structure of the two-dimensional metric  $F = c\beta + \beta^m \alpha^{1-m}$  can be determined if  $F$  is locally projectively flat with  $m \neq 0, \pm 1$  and  $c = 0$  if  $m = -3$ .

*Proof of Theorem 1.3 :*

Let  $F = c\beta + \beta^m \alpha^{1-m}$  be a two-dimensional Douglas  $(\alpha, \beta)$ -metric, where  $c, m$  are constant with  $m \neq 0, \pm 1$  and  $c = 0$  if  $m = -3$ . Then by Theorem 1.2(iii) and (iv),  $F$  can be written as

$$F = c\eta\tilde{\beta} + \tilde{\beta}^m \tilde{\alpha}^{1-m},$$

where  $\eta = \eta(x^1)$  and  $\tilde{\alpha}, \tilde{\beta}$  are given by (8). Now we can easily verify that (12) holds. So  $F$  is projectively flat with  $G^i = Py^i$ . Further, by (13) we can get the projective factor  $P$  given by

$$P = \frac{c\eta_1}{2F}(y^1)^2, \quad \eta_1 := \eta_{x^1}. \quad (102)$$

Besides, its scalar flag curvature  $K$  is given by

$$K = \frac{c(y^1)^3}{2F^3} \left\{ \frac{3c\eta_1^2 y^1}{2F} - \eta_{11} \right\}, \quad \eta_{11} := \eta_{x^1 x^1}. \quad (103)$$

Then by (102) and (103),  $F$  is Berwaldian, or locally Minkowskian if and only if  $c = 0$  or  $\eta = \text{constant}$ . Q.E.D.

Note that the method applied in Theorem 1.2 cannot be used to determine the local structure of the metric  $F = c\beta + \alpha^2/\beta$ , or  $F = c\beta + \alpha^4/\beta^3$  ( $c \neq 0$ ) when  $F$  is locally projectively flat. In this case, we can only obtain a general characterization by Theorem 5.1, as shown in the following corollary.

**Corollary 7.1** *Let  $F$  be a two-dimensional  $(\alpha, \beta)$ -metric. If  $F = c\beta + \alpha^2/\beta$ , then  $F$  is locally projectively flat if and only if the spray  $G_\alpha^i$  of  $\alpha$  satisfy*

$$G_\alpha^i = \rho y^i - \frac{r_{00}}{2b^2} b^i - \frac{\alpha^2}{2b^2} s^i.$$

*If  $F = c\beta + \alpha^4/\beta^3$ , then  $F$  is locally projectively flat if and only if  $\beta$  satisfies (87) and the spray  $G_\alpha^i$  of  $\alpha$  satisfy*

$$G_\alpha^i = \rho y^i + 3\tau\alpha^2 b^i + \left\{ \frac{c}{8}(3b^2\alpha^2 - \beta^2) - \frac{3\alpha^2}{4b^2} \right\} s^i.$$

## 8 Examples

In this section, we will construct some examples which are Douglasian or projectively flat. Further, we show for the metric  $F = c\beta + \alpha^2/\beta$ , or  $F = c\beta + \alpha^4/\beta^3$  ( $c \neq 0$ ), there are examples which are Douglasian but not locally projectively flat.

**Example 8.1** *In Remark 6.3, put*

$$u := x^1, \quad v := x^2, \quad \eta := |x|^{1-m}.$$

*It is easy to see that  $u, v, \eta$  satisfy (101), and  $\alpha$  and  $\beta$  determined by (9) and (100) are given by*

$$\alpha := \frac{|y|}{|x|^{m+1}}, \quad \beta := \frac{\langle x, y \rangle}{|x|^{m+1}}.$$

*Then the  $(\alpha, \beta)$ -metric  $F = c\beta + \beta^m \alpha^{1-m}$  is locally projectively flat, where  $c, m$  are constant with  $m \neq 0, 1$ . But we do not have  $G^i = P y^i$  in the present coordinate system.*

**Example 8.2** *In Remark 6.3, put*

$$u := x^2, \quad v := -x^1, \quad \eta := |x|^{1-m}.$$

*It is easy to verify that  $u, v, \eta$  does not satisfy the third equation in (101), and  $\alpha$  and  $\beta$  determined by (9) and (100) are given by*

$$\alpha := \frac{1}{|x|^{m+1}} |y|, \quad \beta := \frac{1}{|x|^{m+1}} (x^2 y^1 - x^1 y^2).$$

*Then the  $(\alpha, \beta)$ -metric  $F = \beta^m \alpha^{1-m}$  is locally Minkowskian, where  $c, m$  are constant with  $m \neq 0, 1$ . Obviously,  $\beta$  is not closed.*

Now we consider the metrics  $F = c\beta + \alpha^2/\beta$ , and  $F = c\beta + \alpha^4/\beta^3$ . To verify the following two examples, we need to mention the so-called  $K$ -curvature ([9]). For an  $n$ -dimensional

Finsler metric  $F$ , let  $R^i_k$  be the Riemann curvature of  $F$ . Then the  $h$ -curvature tensor  $H_j^i{}_{kl}$  of Berwald connection are defined by

$$H_j^i{}_{kl} := \frac{1}{3} \left( \frac{\partial^2 R^i_l}{\partial y^j \partial y^k} - \frac{\partial^2 R^i_k}{\partial y^j \partial y^l} \right).$$

Further we define

$$H_{jk} := H_j^p{}_{kp}, \quad H_j := \frac{1}{n-1} (nH_{0j} + H_{j0}),$$

and then the coefficients  $K_{ij}$  of the  $K$ -curvature are define as

$$K_{ij} := H_{i;j} - H_{j;i}, \quad (104)$$

where the symbol  $_{;j}$  denotes the  $h$ -derivative of Berwald connection. It is shown in [9] that if a Finsler metric is of scalar flag curvature  $\lambda$ , then we have

$$H_i = (n+1) \left( \frac{1}{3} F^2 \lambda_{y^i} + \lambda F F_{y^i} \right). \quad (105)$$

In [9], it proves that a two-dimensional Finsler metric  $F$  is locally projectively flat if and only if  $F$  is Douglasian and the  $K$ -curvature vanishes  $K_{12} = 0$ .

**Example 8.3** Let  $F = c\beta + \alpha^2/\beta$ , where  $c$  is a constant. Define

$$\alpha = \eta^{\frac{m}{m-1}} \sqrt{(y^1)^2 + (y^2)^2}, \quad \beta = \eta y^1,$$

where  $\eta = \eta(x^2)$ . Then  $F$  is locally projectively flat if and only if  $c\eta''' = 0$ .

*Proof :* By (104) and (105), a direct computation gives

$$K_{12} = -\frac{3}{2} c\eta''' y^1.$$

Now it is clear that  $K_{12} = 0$  if and only if  $c\eta''' = 0$ .

Q.E.D.

**Example 8.4** Let  $F = c\beta + \alpha^4/\beta^3$ , where  $c \neq 0$  is a constant. In (5) and (6), put

$$u = x^2, \quad v = -x^1, \quad B = x^1, \quad c = 1,$$

and let  $\alpha$  and  $\beta$  be defined by (5) and (6). Then by Theorem 1.2(ii),  $F$  is a Douglas metric. However,  $F$  is not locally projectively flat.

*Proof :* Similarly as in the proof to Example 8.3, we only need to compute  $K_{12}$ . A direct computation gives

$$K_{12} = \frac{3(A_1 y^1 + A_2 y^2)}{[4 + 3(x^1)^2]^5 [(x^1)^2 + (x^2)^2]^2},$$

where  $A_1, A_2$  are defined by

$$\begin{aligned} A_1 : &= d\{1296d^7e + e(3555 + 540e^2)d^5 + e(720e^2 + 2820)d^3 + e(224 - 960e^2)d - 1280e^3\}, \\ A_2 : &= d\{-540d^8 + (216e^2 - 2115)d^6 + (720e^2 - 3012)d^4 + (768e^2 - 1248)d^2 + 256e^2\}, \end{aligned}$$

where  $d := x^1, e := x^2$ . Now it is clear that  $K_{12} \neq 0$ . So  $F$  is not locally projectively flat.

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